JANUARY 2009 ANALYSIS QUALIFYING EXAM

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1. Problem 1

(a). Consider

$$A := \{n \in \mathbb{N} \mid n \geqslant 2\}$$

$$B := \{n + 1/n \mid n \in \mathbb{N},\ n \geqslant 2\}$$

These are both obviously closed and disjoint, however,

$$d(A, B) = \inf\{1/n \mid n \geqslant 2\} = 0$$

(b). By definition of infimum, for each $n \in \mathbb{N}$, there exists $(a_n, b_n) \in A \times B$ such that

$$\rho(a_n, b_n) < d(A, B) + 1/n$$

As B is compact, b_n has a convergent subsequence $b_{n_k} \to b \in B$. If $d(A, B) = 0$, by construction we have that

$$d(a_{n_k}, b_{n_k}) \to 0$$

in which case $b \in \overline{A}$, and, as A is closed, $b \in A$, contradicting the fact that A and B are disjoint. Thus $d(A, B) > 0$.

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2. Problem 2

(a). Set $f(t) := \frac{t^p}{p} + \frac{1}{p'} - 1$. We see that $f'(t) = t^{p-1} - 1$ has a real root at $t = 1$ and is positive for $t > 1$, so that t is a minimum. Plugging in $t = 1$,

$$f(1) = 0$$

So that we may take $t = \frac{a}{b^{\frac{1}{p-1}}}$ to see:

$$\begin{aligned} f\left(\frac{a}{b^{\frac{1}{p-1}}}\right) &= \frac{a^p}{pb^{\frac{p}{p-1}}} + \frac{1}{p'} - \frac{a}{b^{\frac{1}{p-1}}} \geqslant 0 \\ \implies \frac{a^p}{p} &+ \frac{b^{p'}}{p'} \geqslant ab \end{aligned}$$

With equality if and only if $a = b^{\frac{1}{p-1}}$.

(b). Now, without loss of generality we may assume (by homogeneity)

$$\Big(\sum_{k=1}^{\infty} |x_k|^p\Big)^{1/p} = \Big(\sum_{k=1}^{\infty} |y_k|^q\Big)^{1/q} = 1$$

Using Young's inequality (as proved above):

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k y_k| &\leqslant \sum_{k=1}^{\infty} \left(\frac{|x_k|^p}{p} + \frac{|y_k|^q}{q}\right) \\ &= \frac{1}{p}\sum_{k=1}^{\infty} |x_k|^p + \frac{1}{q}\sum_{k=1}^{\infty} |y_k|^q \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Yielding Hölder's inequality.

(c). Mimicking the previous part, we may again assume that $||f||_p = ||g||_q = 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\begin{aligned} \int_X |fg| d\mu &\leqslant \frac{||f||_p^p}{p} + \frac{||g||_q^q}{q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Which yields Hölder's inequality.

3. Problem 3

(a). Recall the construction of the Cantor set $C = \bigcap_{n\geqslant 1} C_n$, where each C_n is the nth iteration of removing the middle thirds. Then, by definition we have

$$\mu(C_n) = \left(\frac{2}{3}\right)^n$$

Lettng $n \to \infty$,

$$\mu(C) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

(b). Note that $1/4 = 0.0202020\ldots$ in ternary, in which case $1/4 \in C$ by definition. Since this decimal expansion never terminates, however, $1/4$ cannot be the endpoint of any of the C_n.

(c). Simply note that C is in bijection with $\{0,2\}^{\mathbb{N}}$, since the Cantor set consists of all numbers whose ternary expansion consists only of the numbers 0 and 2. As $\{0,2\}^{\mathbb{N}}$ is uncountable, C is also uncountable.

4. Problem 4

(a). f is absolutely continuous if for all $\epsilon > 0$ there exists δ such that for all sets of open intervals $\{(a_k, b_k)\}$ with

$$\sum_{k=1}^{N} b_k - a_k < \delta$$

we have that

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \epsilon$$

(b). Let $\epsilon > 0$. Assume that g is monotone increasing. Then, there exists δ_1 such that for all open intervals $\{(c_k, d_k)\}$ in (c, d) with

$$\sum_{k=1}^{N} d_k - c_k$$

we have

$$\sum_{k=1}^{N} |f(d_k) - f(c_k)| < \epsilon$$

Similarly, as g is absolutely continuous there exists δ_2 such that for all open intervals $\{(a_k, b_k\}$ with

$$\sum_{k=1}^{N} b_k - a_k < \delta_2$$

we have

$$\sum_{k=1}^{N} |g(b_k) - g(a_k)| < \delta_1$$

Since g is monotonically increasing,

$$|g(b_k) - g(a_k)| = g(b_k) - g(a_k)$$

Whence the set of intervals $\{(g(a_k), g(b_k))\}$ satisfies

$$\sum_{k=1}^{N} g(b_k) - g(a_k) < \delta_1$$

so that

$$\sum_{k=1}^{N} |f(g(b_k)) - f(g(a_k))| < \epsilon$$

Implying that $f \circ g$ is absolutely continuous.

5. Problem 5

We first establish the claim for characteristic functions on some open interval. We see:

$$\begin{aligned}\int_{\mathbb{R}} \chi_{(a,b)} \cos(nx) dx &= \int_a^b \cos(nx) dx \\ &= \frac{\cos(nb) - \cos(na)}{n} \\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

Similarly, consider a step function

$$s = \sum_{k=1}^{N} c_k \chi_{(a_k, b_k)}$$

We see

$$\int_{\mathbb{R}} s\cos(nx)dx = \sum_{k=1}^{N}\int_{a_k}^{b_k}\cos(nx)dx$$

$$\frac{\sum_{k=1}^{N}\cos(nb_k)-\cos(na_k)}{n}$$

$$\to 0 \text{ as } n\to\infty$$

Now let $\epsilon > 0$. By density of step functions in L^1, there exists a step function s with

$$||f-s||_1 < \frac{\epsilon}{2}$$

Similarly, by the above, there exists $N\in\mathbb{N}$ such that for all $n>N$,

$$\int_{\mathbb{R}} g\cos(nx)dx < \epsilon/2$$

$$\Big|\int_{\mathbb{R}} f(x)\cos(nx)dx\Big| \leqslant \int_{\mathbb{R}}|f-g|dx + \Big|\int_{\mathbb{R}} g(x)\cos(nx)dx\Big|$$
$$< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

Whence the result follows immediately.

6. Problem 6

Note that the inequality

$$\sup_{x\in\mathbb{R}} |f*g(x)| \leqslant ||f||_p||g||_q$$

implies the existence of $f*g(x)$ for all x since it is finite everywhere.

By Hölder's inequality,

$$\Big|\int_{\mathbb{R}} f(x-y)g(y)dy\Big| \leqslant \Big(\int_{\mathbb{R}}|f(x-y)|^p dy\Big)^{1/p}\Big(\int_{\mathbb{R}}|g(y)|^q dz\Big)^{1/q}$$
$$= ||f||_p||g||_q$$

Taking the supremum over all x,

$$\sup_{x\in\mathbb{R}} |f*g(x)| \leqslant ||f||_p||g||_q$$

as contended.

7. Problem 7

(a). We have

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

(b). Assume $|f(z)| \leqslant M$. By holomorphicicty, we have a power series expansion

$$f(z) = \sum_{n\geqslant 0} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i}\int_{B_r(0)} \frac{f(z)}{z^{n+1}}dz$$

Consider now for $n \geqslant 1$,

$$\begin{aligned}
|a_n| &\leqslant \frac{1}{2\pi}\int_{B_r(0)} \frac{|f(z)|}{|z|^{n+1}}dz \\
&= \frac{1}{2\pi r^{n+1}}\int_{B_r(0)} |f(z)|dz \\
&\leqslant \frac{1}{2\pi r^{n+1}} \cdot M \cdot 2\pi r \\
&= \frac{M}{r^n}
\end{aligned}$$

As f is entire, we may take $r \to \infty$ to find that $|a_n| = 0$ for all $n \geqslant 1$; that is, $f \equiv a_0$, so that f is constant.

8. Problem 8

Assume $n > 0$ and p is monic. Then, if p has no roots, $1/p(z)$ is entire as the denominator never vanishes.

Setting $M := \sum_{i=0}^n |a_i|$,

$$\begin{aligned}
|P(z)| &\geqslant |z|^n - (M-1)|z|^{n-1} \\
&= |z|^{n-1}(|z| - M + 1)
\end{aligned}$$

In which case, for $|z| \geqslant M$,

$$\frac{1}{|P(z)|} \leqslant \frac{1}{M^{n-1}}$$

Similarly, for $|z| \leqslant M$, we see that by compactness of the closed ball of radius M, $|P(z)|$ achieves its minimum. If this minimum were 0, then $P(z) = 0$, contradicting our assumption. Thus, $P(z) \geqslant m > 0$ for $|z| \leqslant M$. This then implies that $1/P(z)$ is bounded everywhere, and by Louville's theorem, constant. Since we assumed $n > 0$, this is a contradiction.

Then, $P(z)$ has at least one root, say z_1. Now, continue inductively and apply the above argument to $\frac{P(z)}{(z-z_1)}$ to see that P must have precisely n zeroes, counting multiplicity.

The next questions came from a second version of this qualifying exam; the rest of the questions were the same.

9. Problem 6

Let $f(z)$ and $g(z)$ be analytic within and on a simple closed contour C with $|g(z)| < |f(z)|$ on C. Assume f does not vanish on C. Then, $f(z)$ and $f(z) + g(z)$ have the same number of zeroes inside C.

Now, set $f(z) := z^5$ and $g(z) := 3z + 1$. Then, on the boundary of the disk of radius 2,

$$\begin{aligned} |f(z)| &= 32 \\ &> 7 \\ &\geqslant |3z + 1| = |g(z)| \end{aligned}$$

Obviously f has 5 zeroes within $B_2(0)$, so that employing Rouché's theorem, we deduce that $f(z) + g(z) = z^5 + 3z + 1$ has 5 zeroes within $B_2(0)$ as well.

10. Problem 8

Make the change of variable

$$z = e^{i\theta}, \quad \frac{dz}{iz} = d\theta$$

$$\sin^2(\theta) = -\frac{(z^2-1)^2}{4z^2}, \quad \cos(z) = \frac{(z^2+1)}{2z}$$

And,

$$\int_0^{2\pi} \frac{\sin^2(\theta)}{5+4\cos(\theta)} d\theta = \int_{B_1(0)} \frac{i(z^2-1)^2}{4z^2(2z^2+5z+2)} dz$$

Then, $2z^2 + 5z + 2$ has zeroes at -2 and $-1/2$; only $-1/2$ lies within our contour, so we compute the residues at 0 and $-1/2$. We find

$$\text{Res}\Big(\frac{i(z^2-1)^2}{4z^2(2z^2+5z+2)}, 0\Big) = \frac{-5}{16}$$

$$\text{Res}\Big(\frac{i(z^2-1)^2}{4z^2(2z^2+5z+2)}, \frac{-1}{2}\Big) = \frac{3}{16}$$

Whence by Cauchy's Residue theorem,

$$\int_{B_1(0)} \frac{i(z^2-1)^2}{4z^2(2z^2+5z+2)} dz = \frac{\pi}{4}$$

so that

$$\int_0^{2\pi} \frac{\sin^2(\theta)}{5+4\cos(\theta)} d\theta = \frac{\pi}{4}$$